

## International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Website: [www.ijirset.com](http://www.ijirset.com)

Vol. 6, Issue 6, June 2017

# Single Mode Approach of Geometrically Nonlinear Vibration of Beams Carrying a Point Mass and Restrained at the Ends

Adri Ahmed<sup>1</sup>, Benamar Rhali<sup>2</sup>

Professor, Department of Mechanical Engineering, Superior School of Technology, Hassan II University in  
Casablanca, MOROCCO<sup>1</sup>

Professor, Department of Mechanical Engineering, Mohammadia School of Engineers, Mohammed V University in  
Rabat, MOROCCO<sup>2</sup>

**ABSTRACT:**The non-linear vibrations of a beam carrying a point mass at an arbitrary location and restrained by translational and rotational springs at the two ends are investigated analytically and a parametric study is performed, allowing examination of all possible combinations of classical restrained end conditions, including elastic restraints. The dynamic equation was written at two intervals of the beam span with appropriate end and continuity conditions. After the necessary algebraic transformations, the generalised transcendental frequency equation was solved iteratively using the Newton Raphson method. Once the corresponding program implemented, investigations have been made of the changes in the beam frequencies and mode shapes for many values of the mass, mass location and spring stiffness. Numerical results and plots have been given here of the beam frequencies and first mode shape corresponding to various situations. The effect of geometrical non-linearity has then been investigated using a semi analytical method based on Hamilton's principle and spectral analysis leading to solution of a non-linear amplitude equation. A single mode approach, performed in the modal basis, has been adopted in order to obtain, for various configurations of the beam examined, the backbone curves giving the amplitude dependent nonlinear frequencies.

**KEYWORDS:**Nonlinear vibration, Hamilton's principle, spectral analysis, single mode approach, restrained beams.

### I. INTRODUCTION

The operation of machines (machine tools, automotive, robot manipulators and others) introduces dynamic constraints on the various components of the engine and the supporting elements. To ensure correct operation, reduce the induced noise, and increase the machine fatigue life, it is essential to determine the natural frequencies of the system, in both the linear and non-linear regimes. Many such situations may be modelled by a beam carrying one or many masses, restrained at its ends by flexible rotational and translational springs [1-2]. This makes it possible to study different types of restrained end conditions, such as simply supported or clamped, depending on the values assigned to the spring stiffness. The vibrations of restrained beams supporting point masses have been partially examined before [1-3] but all of the studies available are restricted to the linear case. The present paper is based on a systematic parametric study allowing easy choice of the position of the mass to be added in order to adapt the linear frequencies and avoid possible resonances. In the second part, the effect of geometrical nonlinearity on the system "beam + added mass" amplitude dependent nonlinear frequencies is investigated. A single mode approach is adopted, combining the semi analytical method for nonlinear structural vibrations developed previously [4] and the linear modes calculated in the first part, and allowing various backbone curves to be drawn, corresponding to various values of the spring stiffness and added mass.

# International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Website: [www.ijirset.com](http://www.ijirset.com)

Vol. 6, Issue 6, June 2017

## II. LITERATURE REVIEW

Many researchers have worked on the linear and nonlinear vibrations of beams subjected to different boundary conditions. Different theoretical approaches have been used, Unfortunately the results obtained do not really coincide those obtained by the experimentation. This is due mainly to the fact that it is difficult to reach boundary conditions, such as simply supported or clamped beam ends, meeting the requirements of theory. The modeling of the supports by rotational and translational spring makes it possible to bring the theory to the experimentation and to explain the difference between them. Arena and Gossi [11] have presented an approximate solution of the linear vibrations of a uniform beam, with one end spring-hinged and a rotational restraint in a variable position. Rutemberg [29] presented eigenfrequencies for a uniform cantilever beam with a rotational restraint at some position. Lau [18] extended Rutemberg's results including an additional spring to a translation. Nallim and Grossi [12] studied the dynamic behavior of beams such as nonuniform cross sections, and ends elastically restrained against rotation and translation. Rao and Mirza [15] have presented the exact solution of the frequency and normal mode-form expressions for uniform beams with ends elastically restrained against rotation and translation. Grossi [16] showed the role of natural boundary conditions in the Ritz method. Laura and al [19] studied the transverse vibrations of a beam elastically restrained at one end and with a mass and spring at the other subjected to an axial force. Laura and Gutierrez [24] were interested in Vibration of an elastically restrained cantilever beam of varying cross section with tip mass of finite length. In the present work, we are interested in the study of geometrically nonlinear vibrations of Euler-Bernoulli beams carrying a point mass at various position and restrained by translational and rotational springs at the ends.

## III. VIBRATION OF A RESTRAINED BEAM CARRYING A POINT MASS

The uniform beam, with a point mass  $m$ , of Figure.1 is made of a material of mass density  $\rho$ , Young's modulus  $E$ , length  $L$ , cross-sectional area  $S$ , radius of gyration  $r$  and second moment of area of cross section  $I$ . The beam is restrained at the ends by translational and rotational springs. Let  $x$  be the coordinate along the neutral axis of the beam measured from the right end. The beam transverse displacement can be written as [1]  $W(x,t) = w(x) \sin(\omega t)$ , measured from its equilibrium position.

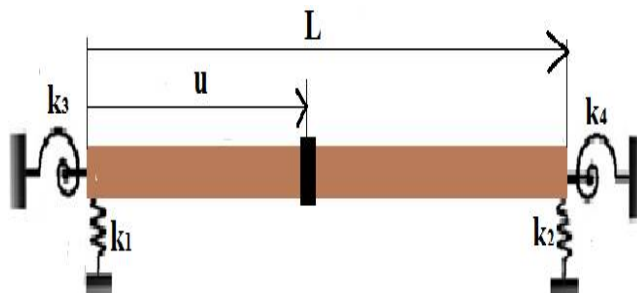


Figure. 1. The restrained beam with a point mass. Beam characteristics ( $L, S, \rho, E, I$ )

The problem under consideration is governed by the following differential equation:

$$\frac{d^4 w}{dx^4} - \beta^4 w = 0 \text{ with } \beta^4 = \frac{\rho S}{EI} \omega^2 \quad (1)$$

The function  $w$  is defined in piecewise by:  $w_1(\eta)$  and  $w_2(\eta)$  in  $]0, \xi[$  and  $]\xi, 1[$  respectively, with  $\eta = \frac{x}{L}$ ;  $\xi = \frac{u}{L}$ .

The general solution for transverse vibration in the first and second span, can be written as:

$$w_1(\eta) = a_1 \cosh(\beta_1 L \eta) + b_1 \sinh(\beta_1 L \eta) + c_1 \cos(\beta_1 L \eta) + d_1 \sin(\beta_1 L \eta) \quad (2)$$

## International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Website: [www.ijirset.com](http://www.ijirset.com)

Vol. 6, Issue 6, June 2017

$$w_2(\eta) = a_2 \cosh(\beta_i L(\eta - \xi)) + b_2 \sinh(\beta_i L(\eta - \xi)) + c_2 \cos(\beta_i L(\eta - \xi)) + d_2 \sin(\beta_i L(\eta - \xi)) \quad (3)$$

In which  $\beta_i = \sqrt[4]{\frac{\rho S \omega_i^2}{EI}}$  for  $i = 1, 2, \dots$  are the mode shape parameters of the beam with an added point mass. The constants  $a_j, b_j, c_j, d_j$  are determined by the continuity and end conditions:

At the ends:

$$\left. \frac{d^3 w_{1i}(\eta)}{d\eta^3} \right|_{\eta=0} = -k_1 w_{1i}(\eta) \Big|_{\eta=0} \quad (4)$$

$$\left. \frac{d^2 w_{1i}(\eta)}{d\eta^2} \right|_{\eta=0} = k_3 \left. \frac{dw_{1i}(\eta)}{d\eta} \right|_{\eta=0} \quad (5)$$

Continuity conditions:

$$\left. \frac{d^3 w_{(n+1)i}(\eta)}{d\eta^3} \right|_{\eta=1} = k_2 w_{(n+1)i}(\eta) \Big|_{\eta=1} \quad (6)$$

$$\left. \frac{d^2 w_{(n+1)i}(\eta)}{d\eta^2} \right|_{\eta=1} = -k_4 \left. \frac{dw_{(n+1)i}(\eta)}{d\eta} \right|_{\eta=1} \quad (7)$$

$$w_{2i}(\eta) \Big|_{\eta=\xi} = w_{1i}(\eta) \Big|_{\eta=\xi} \quad (8)$$

$$\left. \frac{dw_{2i}}{d\eta} \right|_{\eta=\xi} = \left. \frac{dw_{1i}}{d\eta} \right|_{\eta=\xi} \quad (9)$$

$$\left. \frac{d^2 w_{2i}}{d\eta^2} \right|_{\eta=\xi} = \left. \frac{d^2 w_{1i}}{d\eta^2} \right|_{\eta=\xi} \quad (10)$$

$$\left. \frac{d^3 w_{2i}}{d\eta^3} \right|_{\eta=\xi} = \left. \frac{d^3 w_{1i}}{d\eta^3} \right|_{\eta=\xi} + M (\beta_i L)^4 w_{1i}(\eta) \Big|_{\eta=\xi} \quad (11)$$

Where  $M = \frac{m}{\rho SL}$

Equations 4 to 11 give a linear system with eight equations and eight unknowns whose determinant must vanish, leading via application of a Newton–Raphson algorithm, to the vibrating beam frequencies and mode shapes. The corresponding numerical results are summarised in Table1.

# International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Website: [www.ijirset.com](http://www.ijirset.com)

Vol. 6, Issue 6, June 2017

Table1: eigenvalues of the “beam mass” for different values of the rotational stiffness, mass and mass locations

$\eta$	$K_3=10 \quad k_4=10$		$K_3=100 \quad k_4=100$		$K_3=10 \quad k_4=100$	
	$M=0,5$	$M=1$	$M=0,5$	$M=1$	$M=0,5$	$M=1$
0	4,15566	4,15566	4,64132	4,64132	4,39001	4,39001
0,1	4,10204	4,04572	4,61416	4,58502	4,31999	4,24643
0,2	3,89071	3,66687	4,40167	4,17988	4,06073	3,79719
0,3	3,64396	3,32802	4,07685	3,72112	3,78953	3,44078
0,4	3,48471	3,13988	3,85720	3,46419	3,63523	3,26359
0,5	3,43214	3,08101	3,78481	3,38463	3,60750	3,23319

## IV. APPLICATION: A UNIFORM RESTRAINED BEAM WITH ONE POINT MASS

The effect of the added mass location on the beam first frequency, with its associated mode and curvatures, is shown in Figure 2 for various values of the rotational stiffness for  $M = 0.5$ , located at  $u=L/2$ . These results are summarised in Table 1. Figure 2(a) shows that the mode shape changes slightly as a function of the value of the rotational spring stiffness. Figure 2 (b) shows that the curvature, at the beam ends, increases with the increase of the rotational spring stiffness.

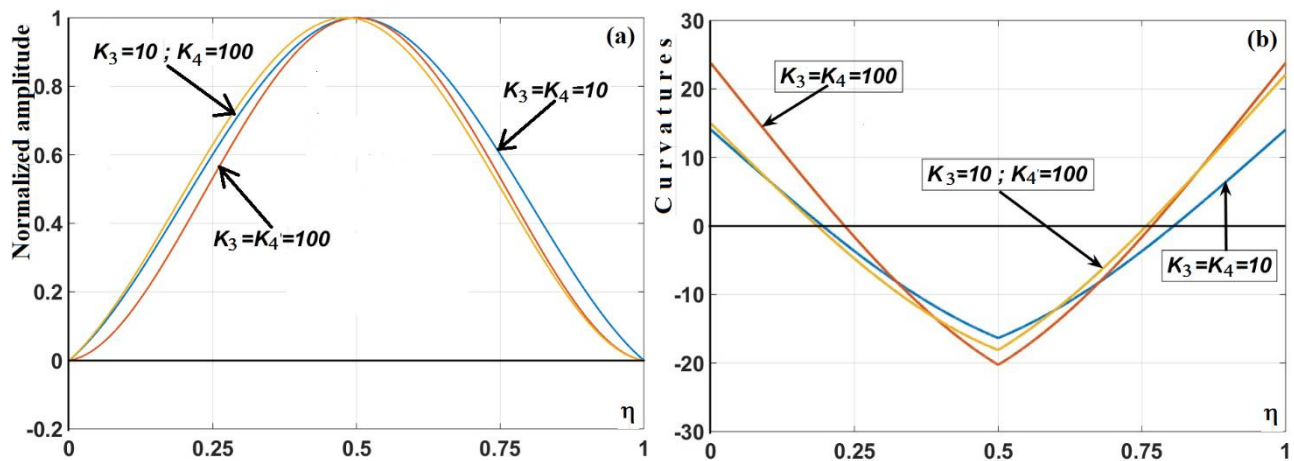


Figure. 2 (a) First mode and (b) Curvatures for  $M = 0.5, u=0.5$  and various values of  $k$

## V. GEOMETRICALLY NONLINEAR VIBRATION OF A RESTRAINED BEAM CARRYING A CONCENTRATED MASS.

At large vibration amplitudes, the beam shown in Figure 1 kinetic energy  $T$ , linear strain energy  $V_{lin}$  and nonlinear strain energy  $V_{Nlin}$  induced by large deflections can be expressed as [4]:

$$T = \frac{1}{2} \rho S \int_0^L \left( \frac{\partial w(x,t)}{\partial t} \right)^2 dx + \frac{1}{2} m \left( \frac{\partial w(x_j,t)}{\partial t} \right)^2 \quad (12)$$

# International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Website: [www.ijirset.com](http://www.ijirset.com)

Vol. 6, Issue 6, June 2017

$$V_{Lin} = \frac{1}{2} EI \int_0^L \left( \frac{\partial^2 w(x,t)}{\partial x^2} \right)^2 dx \tag{13}$$

$$V_{Nlin} = \frac{1}{8} \frac{ES}{L} \left[ \int_0^L \left( \frac{\partial w(x,t)}{\partial x} \right)^2 dx \right]^2 \tag{14}$$

Expanding  $w(x,t)$  as a series of basic spatial functions:  $w(x,t) = q_i(t)w_i(x) = a_i w_i \sin(\omega t)$  and applying Hamilton's principle and integrating the time functions over a period of vibration, the system dynamics is governed by [1]:

$$2[K]\{A\} + 3[B(\{A\})]\{A\} = 2\omega^2 [M]\{A\} \tag{15}$$

in which  $\{A\}$  is the column vector of the basic function coefficients, and  $[K]$  and  $[M]$  are the rigidity and mass matrices, and  $[B(\{A\})]$  is the nonlinear geometrical rigidity. Equation 15 is the Rayleigh-Ritz formulation of the nonlinear problem, to be solved numerically, or explicitly. From equation 15, the frequency  $\omega$  may be obtained by pre-multiplying the two hand sides of the equation by  $\{A\}^T$ , which gives:

$$\omega^2 = \frac{\{A\}^T [K]\{A\} + \frac{3}{2}\{A\}^T [B(\{A\})]\{A\}}{\{A\}^T [M]\{A\}} \tag{16}$$

$$\omega^2 = \frac{k}{m} + \frac{3}{2} a^2 \frac{b}{m} \tag{17}$$

The single mode approach, consists of neglecting all the basic functions except a single 'resonant' mode. Thus, it reduces equation 16 to 17, in which  $[K]=k_{11}$ ,  $[M]=m_{11}$ ,  $[B(\{A\})]=b_{1111}$ . Figure 3(a) shows the backbone curves corresponding to various values of the mass, mass location and rotational spring stiffness. Figure 3(b) shows the curvatures associated to the first nonlinear mode for various vibrations level.

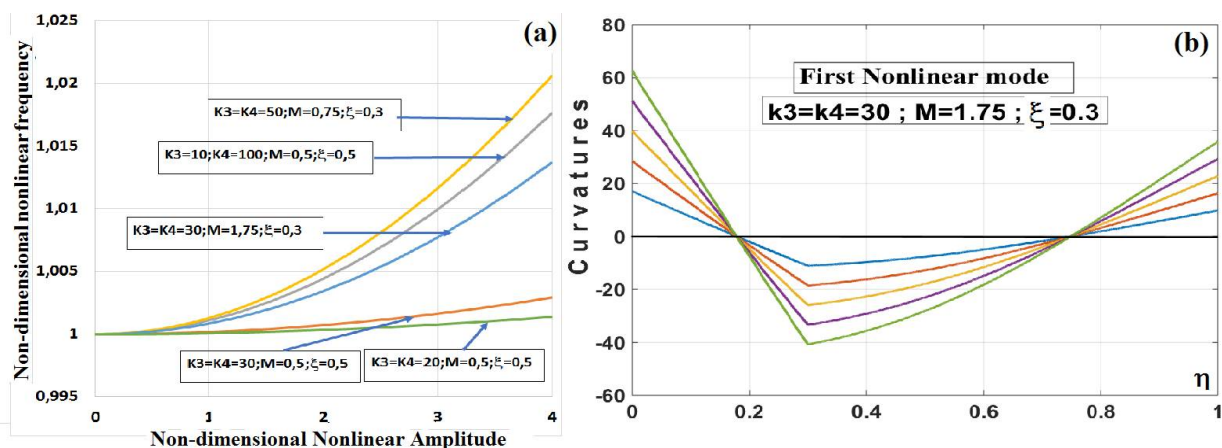


Figure 3 (a) Backbone curves for various values of the mass, mass location and rotational stiffness; (b) curvatures associated to the first nonlinear mode

# International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Website: [www.ijirset.com](http://www.ijirset.com)

Vol. 6, Issue 6, June 2017

## VI. CONCLUSION

The non-linear vibrations of a beam carrying a point mass at an arbitrary location and supported by translational and rotational springs at the two ends have been investigated analytically and a parametric study was performed, allowing examination of many combinations of classical end conditions, including elastic restraints. The dynamic equation was written at two intervals of the beam span with appropriate end and continuity conditions. After the necessary algebraic transformations, the generalised transcendental frequency equation was solved iteratively using the Newton Raphson method. Numerical results and plots have been given of the beam frequencies and first mode shape corresponding to various situations. The effect of geometrical non-linearity has then been investigated using a semi analytical method based on Hamilton's principle and spectral analysis leading to solution of a non-linear amplitude equation. A single mode approach, performed in the modal basis, has been adopted in order to obtain, for various configurations of the beam examined, the backbone curves giving the amplitude dependent nonlinear frequencies.

## REFERENCES

- [1] Adri A, Beidouri Z, El kadiri M, Benamar R. "Geometrically Nonlinear Free Vibration of a Beam Carrying Concentrated Masses" International. Journal of Engineering Research & Technology (IJERT) ISSN: 2278-0181 Vol. 5 Issue 01, January-2016
- [2] Santiago Maiz et al, "Transverse vibration of Bernoulli–Euler beams carrying point masses and taking into account their rotatory inertia: Exact solution", Journal of Sound and Vibration 303 895–908, 2007
- [3] A. H. REGISTER, "a note on the vibrations of generally restrained end-loaded beams", Journal of Sound and Vibration 172(4), 561-571, 1994
- [4] Azrar R. L ,Benamar R. , White R.G. "A semi-analytical approach to the non-linear dynamic response problem of S-S and C-C beams at the large vibration amplitudes part1: General theory and application to the single mode approach to free and forced vibration analysis". Journal of Sound and Vibration 224 (2), 183-207, 1999
- [5] EL Kadiri M., Benamar R. and White.R.G. "Improvement of the semi-analytical method, based on Hamilton's principle and spectral analysis, for determination of the geometrically nonlinear free response of thin straight structures. Part I: Application to C-C and SS-C beams". Journal Sound and Vibration (2002) 249(2), pp 263-305,2002
- [6] Adri Ahmed, BeidouriZitouni, BenamarGhali, El kadiriMouna "Geometrically Nonlinear Mode Shapes And Resonant Frequencies of Three Span Beams" International conference on structural engineering dynamics ICEDynTavira, Portugal 2011
- [7] Adri Ahmed, BeidouriZitouni, BenamarGhali, El kadiriMouna "Geometrically Nonlinear, Mode Shapes And Resonant Frequencies of multi Span Beams" EUROODYN 2011? 8th International Conference on Structural Dynamics. Leuven, Belgium 4-6 july 2011.
- [8] Benamar,R. , Bennouna M. and White. R. G.,1991. "The effects of large vibration amplitudes on the fundamentalmode shape of thin elastic structures, part I: Simply supported and clamped-clamped beams". Journal of Sound and Vibration 149 pp 179-195, 1991
- [9] Benamar R."Non-linear dynamic behavior of fully clamped beams and rectangular isotropic and laminatedplates" Ph.D. thesis, University of Southampton, 1990
- [10] Bennouna M. and White. R.G."The effects of large vibration amplitude on the fundamental mode shape of a clamped-clamped beam". Journal of sound and vibration, (3) 309-331, 1984
- [11] B. Arenas and R. O. Grossi, "Vibration frequencies for a beam with a rotational restraint in an adjustable position", Applied Acoustics 57, 197–202, 1999
- [12] L. Nallim and R. O. Grossi, "A general algorithm for the study of the dynamical behavior of beams", Applied Acoustics 57, 345–356, 1999
- [13] R. O. Grossi, A. Aranda, and R. B. Bhat, "Vibration of tapered beams with one end spring hinged and the other end with tip mass", J. Sound Vibration 160 ,175–178, 1993
- [14] R. O. Grossi and R. B. Bhat, "A note on vibrating tapered beams", J. Sound Vibration 147, 174–178, 1991
- [15] C. K. Rao and S. Mirza, "Note on vibrations of generally restrained beams", J. Sound Vibration 130, 453–465, 1989
- [16] R. O. Grossi, "On the role of natural boundary conditions in the Ritz method", International Journal of Mechanical Engineering Education 16 , 207–210, 1988
- [17] H. Cortinez and P.A.A. Laura, "Vibration and buckling of a non-uniform beam elastically restrained against rotation at one end and with concentrated mass at the other", J. Sound Vibration 99, 144–148, 1985
- [18] J. H. Lau, "Vibration frequencies and mode shapes for a constrained cantilever", Journal of Applied Mechanics 57, 182–187, 1984
- [19] P. A. A. Laura, R. O. Grossi, and S. Alvarez, "Transverse vibrations of a beam elastically restrained at one end and with a mass and spring at the other subjected to an axial force", Nuclear Engineering and Design 74, 299–302, 1982
- [20] P. Goel, Free "vibrations of a beam-mass system with elastically restrained ends" J. Sound Vibration 47, 9–14, 1976
- [21] A. Grant, "Vibration frequencies for a uniform beam with one end elastically supported and carrying a mass at the other end", Journal of Applied Mechanics 42, 878–880, 1975
- [22] R. O. Grossi and P. A. A. Laura, "Further results on a vibrating beam with a mass and spring at the end subjected to an axial force", J. Sound Vibration 84,593–594. 1982
- [23] C. Hibbeler, "Free vibrations of a beam supported with unsymmetrical spring hinges", Journal of Applied Mechanics 42, 501–502, 1975
- [24] P. A. A. Laura and R. H. Gutierrez, "Vibration of an elastically restrained cantilever beam of varying cross section with tip mass of finite length" J. Sound Vibration 108, 123–131, 1986

## International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Website: [www.ijirset.com](http://www.ijirset.com)

Vol. 6, Issue 6, June 2017

- [25] W. Lee, "Vibration frequencies for a uniform beam with one end spring hinged and carrying a mass at the other free end", Journal of Applied Mechanics **40**, 813–815, 1973
- [26] J. Acoust. Soc. Amer., "Transverse vibrations of double-tapered cantilever beams". **57**, 1771–1774, 1972
- [27] J. Acoust. Soc. Amer., "Transverse vibrations of double-tapered cantilever beams with end support and with end mass", **55**, 986–991, 1974
- [28] R. Maurizi, R. Rossi, and J. Reyes, "Vibration frequencies for a uniform beam with one end spring hinged and subjected to a translational restraint at the other end", J. Sound Vibration **48**, 565–568, 1976
- [29] A. Rutenberg, "Vibration frequencies for a uniform cantilever with a rotational constraint at a point", Journal of Applied Mechanics **45**, 422–423, 1978
- [30] R. Chun, "Free vibration of a beam with one end spring-hinged and the other free", Journal of Applied Mechanics **39**, 1154–1155, 1972
- [31] M. S. Hess, "Vibration frequencies for a uniform beam with central mass and elastic supports", Journal of Applied Mechanics **31**, 556–558, 1964
- [32] K. Rektorys, "Variational Methods in Mathematics, Science and Engineering", D. Reidel, Dordrecht, 1980.
- [33] J. L. Troutman, "Variational Calculus and Optimal Control, Undergraduate Texts in Mathematics", Springer-Verlag, New York, 1996.